Homework #3

1. Mark each statement True or False. Justify your answer.
(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to p as the antecedent.
True, In the implication $p \Rightarrow q$ p: antecedent q: Consequent.
(b) The contrapositive of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$
False, The contrapositive of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$. We are given the inverse $\neg p \Rightarrow \neg q$.
(c) The inverse of $p \Rightarrow q$ is $\neg q \Rightarrow \neg p$
False, The inverse of $p \Rightarrow q$ is $\neg p \Rightarrow \neg q$. We are given the contrapositive $\neg q \Rightarrow \neg p$.
(d) To prove "$\forall$ n, p(n)" is true, it takes only one example.
False, In order to prove "$\forall$ n, p(n)" is true" we need to do a general proof that does not revolve around any specific example.
(e) To prove "$\exists$ n, p(n)" is true, it takes only one example.
True, In order to prove $\exists$ n, p(n)" is true we need to show that there exists at least one $n \in \mathbb{R}$ that makes p(n) true , so one example is enough to prove this is true.

2. Mark each statement True or False. Justify your answer.
(a) When an implication $p \Rightarrow q$ is used as a theorem, we refer to q as the conclusion.
True, When $p \Rightarrow q$ is used as a theorem, p:hypothesis q:conclusion
(b) The converse of $p \Rightarrow q$ is $q \Rightarrow p$
True, the converse of f $p \Rightarrow q$ is $q \Rightarrow p$.
(c) To prove "$\forall$ n, p(n)" is false, it takes only one counterexample.
True, If "$\forall$ n, p(n)" was true, it would be true for all n, but since this statement is false we will negate the "$\forall$ n, p(n)" statement. The negation is $\exists$ n, $\neq$p(n)", so in order to prove the negation you can find at least one example that makes the statement false.
(d) To prove "$\exists$ n, p(n)" is false, it takes only one counterexample.
False, To prove "$\exists$ n, p(n)" is false you will take the negation of "$\exists$ n, p(n)" statement. The negation will therefore be "$\forall$ n, $\neq$p(n)", In order to prove the negation of the original statement one counterexample will not be enough.

3. Write the contrapositive of each implication.
(a) If roses are red, then some violets are blue.
Implication: p $\Rightarrow$ q
p: All roses are red
q: Some violets are blue
Contrapositive: $\neg$ q $\Rightarrow$ $\neg$ p
If all violets are not blue then some roses are not red.
(b) A is not invertible, if there exists a nontrivial solution to Ax=0.
Implication: q $\Rightarrow$ p
p: There exists a nontrivial solution
q: A is not invertible
Contrapositive: $\neg$ q $\Rightarrow$ $\neg$ p
If A is invertible then there does not exist a nontrivial solution to Ax=0.
(c) If *f* is continuous and C is connected, then *f(C)* is connected.
Implication: p $\Rightarrow$ q
p: f is continuous and c is connected
q: f(c) is connected
Contrapositive: $\neg$ q $\Rightarrow$ $\neg$ p
If f(c) is not connected then f is not continuous or c is not connected.

4. Write the converse of each implication in Exercise 3.
(a) If roses are red, then some violets are blue.
Implication: p $\Rightarrow$ q
p: All roses are red
q: Some violets are blue
Converse: q $\Rightarrow$ p
If some violets are blue, then all roses are red.
(b) A is not invertible, if there exists a nontrivial solution to Ax=0
Implication: q $\Rightarrow$ p
p:There exists a nontrivial solution
q: A is not invertible
Converse: p $\Rightarrow$ q
There exists a nontrivial solution to Ax=0 if A is not invertible.
(c) If *f* is continuous and C is connected, then *f(C)* is connected.
Implication: p $\Rightarrow$ q
p: f is continuous and c is connected
q: f(c) is connected
Converse: q $\Rightarrow$ p
If f(c) is connected then f is continuous and c is connected.

5.Write the inverse of each implication in Exercise 3.

(a) If all roses are red, then some violets are blue. **Implication: p $\Rightarrow$ q**
p: All roses are red
q: Some violets are blue
Inverse: $\neg$ p $\Rightarrow$ $\neg$ q
If some roses are not red then all violets are not blue.
(b) A is not invertible, if there exists a nontrivial solution to Ax=0
Implication: q $\Rightarrow$ p
p: There exists a nontrivial solution
q: A is not invertible
Inverse: $\neg$ p $\Rightarrow$ $\neg$ q
If there does not exists a nontrivial solution to Ax=0 then A is invertible.
(c) If *f* is continuous and C is connected, then *f(C)* is connected.
Implication: p $\Rightarrow$ q
p: f is continuous and c is connected
q: f(c) is connected
If f is not continuous or c is not connected, then f(c) is not connected.

6. Prove a counterexample for each statement.
(a) For every real number x, if $x^2 > 9$ then x > 3
Take x:=-4, $x^2 >$ 9 but x $\not>$ 3
(b) For every integer n, we have $n^3 \geq$ n.
Take n:=-2, $-2^3 \ngeq$ -2
(c) For all real numbers x $\geq$ 0, we have $x^2 \leq x^3$
Take x:=1/2, $x^2 \nleq x^3$
(d) Every triangle is a right triangle.
An equilateral triangle with angles 60 degrees, 60 degrees, 60 degrees is a triangle, but it is not a right triangle.
(e) For every positive integer n, $n^2 + n + 41$ is prime.
Take n:=41 n:41 is a positive integer but $n^2 + n + 41$ is not prime.
(f) Every prime is an odd number.
Take x:=2, x is prime but is not an odd number.
(g) No integer greater than 100 is prime.
Take x:=101, x > than 100 and is prime.
(h) 3^n+2 is prime for all positive integers n.
Take n:=17, $3^{17} + 2 = 129140165$, which is divisible by 5.
(i) For every integer n > 3, 3n is divisible by 6.
Take n:=5, n > 3, but 3n is not divisible by 6.
(j) If x and y are unequal positive integers and xy is perfect square, then x and y are perfect squares.
Take x:=8, Take y:=2, x and y are unequal positive integers and xy is perfect square. but x and y are not perfect squares.

(k) For every real number x, there exists a real numbers such that xy=2

Take x:=0 xy $\neq$ 2

(l) The reciprocal of a real number x $\geq$1 is a real number y such that 0 < y < 1.

Take x:=1, The reciprocal of x is not a real number y that fits these conditions 0 < y < 1.

(m) No rational number satisfies the equation x^3+$(x-1)^2$= x^2+1

Take x:=0, x is a rational number and satisfies the equation x^3+$(x-1)^2$= x^2+1

(n) No rational number satisfies the equation x^4+(1/x)-$\sqrt{x+1}$ =0

Take x:=-1, x is a rational number and satisfies the equation x^4+(1/x)-$\sqrt{x+1}$ =0

7. Prove the following

(a) If p is odd and q is odd, then p+q is even.

Proof:

Suppose p is odd and q is odd

Since p is odd $\exists$ a $\in$ $\mathbb{R}$ such that p=2a+1

Since q is odd $\exists$ b $\in$ $\mathbb{R}$ such that p=2b+1

We want to prove that p+q is even so that means we want to prove that $\exists$ c $\in$ $\mathbb{R}$ such that p+q=2c.

p+q=(2a+1)(2b+1)=2a+2b+2

=2(a+b+1)

Take c:=a+b+1

Therefore, $\exists$ c $\in$ $\mathbb{R}$ such that p+q=2c.

(b) If p is odd and q is odd then pq is odd.

Proof: Suppose p is odd and q is odd.

Since p is odd $\exists$ a $\in$ $\mathbb{R}$ such that p=2a+1

Since q is odd $\exists$ b $\in$ $\mathbb{R}$ such that p=2b+1

We want to prove pq is odd so that means that $\exists$ c $\in$ $\mathbb{R}$ such that pq=2c+1

pq=(2a+1)(2b+1)=4ab^2+2a+2b+1

=2(2ab^2+a+b)+1

Take c:=2ab^2+a+b

Therefore $\exists$ c $\in$ $\mathbb{R}$ such that pq=2c+1

(c) If p is odd and q is odd, then p+3q is even.

Proof: Suppose p is odd and q is odd

Since p is odd $\exists$ a $\in$ $\mathbb{R}$ such that p=2a+1

Since q is odd $\exists$ b $\in$ $\mathbb{R}$ such that p=2b+1

We want to prove that p+3q is even so that means $\exists$ c $\in$ $\mathbb{R}$ such that p+3q=2c

p+3q=2a+3(2b)=2a+6b

=2(a+3b)

Take c:=a+3b

Therefore, $\exists$ c $\in$ $\mathbb{R}$ such that p+3q=2c

(d) If p is odd and q is even, then p+q is odd.
Proof:
Suppose p is odd and q is even
Since p is odd $\exists$ a $\in \mathbb{R}$ such that p=2a+1
Since q is even $\exists$ a $\in \mathbb{R}$ such that q=2b
We want to prove that p+q is odd so that means $\exists$ c $\in \mathbb{R}$ such that p+q=2c+1
p+q=2a+1+2b
= 2(a+b)+1
Take c:=a+b
Therefore, $\exists$ c $\in \mathbb{R}$ such that p+q=2c+1
(e) If p is even and q is even, then p+q is even.
Proof:
Suppose p is even and q is even
Since p is even $\exists$ a $\in \mathbb{R}$ such that p=2a
Since q is even $\exists$ a $\in \mathbb{R}$ such that q=2b
We want to prove that p+q is even so that means $\exists$ c $\in \mathbb{R}$ such that p+q=2c
p+q= 2a+2b=
2(a+b)
Take c:=a+b
Therefore $\exists$ c $\in \mathbb{R}$ such that p+q=2c
(f) If p is even or q is even, then pq is even.
Proof:
Suppose p is even or q is even
If p is even $\exists$ a $\in \mathbb{R}$ such that p=2a
If q is even $\exists$ a $\in \mathbb{R}$ such that q=2b
We want to prove that pq is even that means $\exists$ c $\in \mathbb{R}$ such that pq=2c
pq=(2a)(2b)=4ab
=2(2ab)
Take c:=2ab
Therefore $\exists$ c $\in \mathbb{R}$ such that pq=2c
(g) If pq is odd, then p is odd and q is odd.
Draft:
P:pq is odd
Q: p is odd and q is odd
Implication: P $\Rightarrow$ Q
$\neg$ P:pq is even
$\neg$ Q: p is even or q is even
Contrapositive: $\neg$ Q $\Rightarrow$ $\neg$ P
If p is even or q is even then pq is even.
Proof:
Suppose p is even or q is even

If p is even $\exists$ a $\in \mathbb{R}$ such that p=2a
If q is even $\exists$ b $\in \mathbb{R}$ such that q=2b
We want to prove pq is even so that means $\exists$ c $\in \mathbb{R}$ such that pq=2c
pq=(2a)(2b)=4ab
2(2ab)
Take c:=2ab
Therefore, $\exists$ c $\in \mathbb{R}$ such that pq=2c
(h) If p^2 is even, then p is even.
Draft:
P: p^2 **is even**
Q: p is even
Implication: P $\Rightarrow$ Q
$\neg$ **P:** p^2 **is odd**
$\neg$ **Q: p is odd**
Contrapositive: $\neg$ Q $\Rightarrow$ $\neg$ P
If p is odd then p^2 **is odd.**
Proof:
Suppose p is odd
Since p is odd $\exists$ a $\in \mathbb{R}$ such that p=2a+1
We want to prove p^2 **is odd that means $\exists$ b $\in \mathbb{R}$ such that** p^2 **=2b+1**
p^2**=(2a+1)(2a+1)**
=4a^2**+4a+1**
=2(2a^2**+2a)+1**
Take b:=2a^2**+2a**
Then $\exists$ b $\in \mathbb{R}$ such that p^2 **=2b+1**
(i) If p^2 is odd, then p is odd.
Draft:
P: p^2 **is odd**
Q: p is odd
Implication: P $\Rightarrow$ Q
$\neg$ **P:** p^2 **is even**
$\neg$ **Q: p is even**
Contrapositive: $\neg$ Q $\Rightarrow$ $\neg$ P
If p is even then p^2 **is even**
Proof:
Suppose p is even
Since p is even $\exists$ a $\in \mathbb{R}$ such that p=2a
We want to prove that p^2 **is even so that means $\exists$ b $\in \mathbb{R}$ such that** p^2 **=2b** p^2
=$(2a)^2$ =$4a^2$
=2(2a^2**)**
Take b:=2a^2

Therefore $\exists$ b $\in \mathbb{R}$ such that p^2 =2b

For problems g,h,i one can see that trying to prove $p \Rightarrow q$ does not work. Therefore, it is essential to try and look for alternate ways to prove the statements.